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1998 J. Phys. A: Math. Gen. 31 4541

(http://iopscience.iop.org/0305-4470/31/19/017)

J. Phys. A: Math. Gen. **31** (1998) 4541–4553. Printed in the UK PII: S0305-4470(98)85753-3

Complex measure, coherent state and squeezed state representation

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Received 8 July 1997

Abstract. Coherent states are studied within the framework of complex measurable processes. It is shown that squeezed states can be accommodated within this framework and all the relevant results relating to squeezed states are derived. A generalization to complex coordinates is also provided and it is shown to lead to the Wigner distribution in a natural way.

1. Introduction

The object of this paper is to establish the viability of the complex measure theoretic framework to accommodate the coherent state and squeezed state extensively discussed in the literature (see, for example, Glauber (1963), Sudarshan (1963), Yuen (1976), Loudon and Knight (1989)). Complex measurable processes were introduced earlier (Srinivasan and Sudarshan (1994), Srinivasan (1995, 1997) referred to as papers I, II and III respectively), mainly to describe quantum phenomena by an enlargement of the notions of probability based on positive definite measures. Instead of interpreting the two slit experiment as a symptom of failure of classical notions, it was proposed that a framework of an extended measure can form the basis for a theory of quantum phenomena. In paper II, the harmonic oscillator was discussed in all its details since it forms the basis for the discussion of the general problem of radiation. The discussion of the forced harmonic oscillator was continued in paper III where it was shown that the problem of interaction of radiation with matter could be accommodated within the framework of complex measure and measurable processes. The present contribution is in the same spirit and we show how the coherent state and the squeezed coherent state can be accommodated within the complex measure theoretic framework (CMTF).

The layout of the paper is as follows. In section 2 the displaced oscillator is identified as a Markov process with appropriate drift and diffusion functions. The complex measure density is then arrived at as the solution of the resulting Fokker–Planck equation: the properties of the coherent states are then derived. Section 3 deals with the squeezed coherent state in a similar manner. The next section deals with the complex coordinate representation for an oscillator and it is shown to lead to a Wigner distribution. The final section contains a summary and discussion of the main results of the paper.

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2. Coherent state representation

We first consider the harmonic oscillator in the CMTF. At the outset we note that the basic assumptions are as follows:

(i) the underlying complex measurable stochastic process $\{X(t); t \geqslant 0\}$ is Markov and the measure of any (measurable) set is absolutely bounded;

(ii) the diffusion and drift limits do exist in the sense that

$$\lim_{\Delta\to\infty} \frac{E\{[X(t+\Delta t) - X(t)]^2 \mid X(t) = x\}}{\Delta t} = 2D = \frac{\mathrm{i}\hbar}{M} \tag{2.1}$$

$$\lim_{\Delta\to\infty} \frac{E\{[X(t+\Delta t) - X(t)] \mid X(t) = x\}}{\Delta t} = 2D = -\mathrm{i}\omega x. \tag{2.2}$$

If we introduce the complex measure density (CMD) $\Pi(x \mid x_0; t)$ by

$$\Pi(x \mid x_0; t) = \lim_{\Delta\to\infty} \frac{\Pr\{x < X(t) < x + \Delta \mid X(0) = x_0\}}{\Delta} \tag{2.3}$$

then it is shown in paper II that $\Pi(x \mid x_0; t)$ satisfies a Fokker–Planck equation provided the total measure is constrained to be unity and the solution itself is given by

$$\Pi(x \mid x_0; t) = \left[\exp\left(-\frac{M\omega}{\hbar}\frac{[x\,\mathrm{e}^{\mathrm{i}\omega t} - x_0]^2}{\mathrm{e}^{2\mathrm{i}\omega t} - 1}\right)\right]\left(\frac{M\omega\,\mathrm{e}^{2\mathrm{i}\omega t}}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\right)^{\frac{1}{2}}. \tag{2.4}$$

We can indeed generate a stationary process by receding the time origin to $-\infty$. The stationary complex measure density turns out to be real and can be identified as the limit of Π as $t \to \infty$ (under the gimmick $\omega \to \omega - \mathrm{i}\epsilon$)

$$\Pi(x) = \lim_{t\to\infty} \Pi(x \mid x_0; t) = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega x^2}{\hbar}\right). \tag{2.5}$$

The modulus measure density introduced in paper I is indeed $\Pi(x)$ itself. Incidentally, we have the result that the ground state is the only state that is stationary in the strict probabilistic sense, its probability density being specified by (2.5).

Next we note that the modulus measure introduced here extended to all measurable sets can be shown to lead to a Hilbert space (see paper III). Generalizing the results of Feller (1971), we can introduce the square root g of any complex measure density f and define the norm $\|g\|$ by

$$\begin{aligned} \|g\| &= \int [g(x)]^2 \,\mathrm{d}x \\ &= \int [f(x)] \,\mathrm{d}x < \infty. \end{aligned}$$

The class of all functions g will be denoted by L^2; the norm induces the natural metric (g_1, g_2) and the metric space L^2 is complete with the inner product defined by

$$(g_1, g_2) = \int g_1(x)\overline{g_2(x)}\,\mathrm{d}x$$

so that the space is rendered to be a Hilbert space. Thus, we have a fusion of probability with Hilbert space and we are comfortably placed to answer questions that are raised and answered in the conventional treatment of quantum phenomena.

The measure density given by (2.5) pertains to the coordinate of the harmonic oscillator. To deal with the momentum, we attempt to be in strict conformity with complementarity principles; the duality principle of de Broglie can be subsumed (see, for example, Misner *et al* (1972)) by taking the momentum as the dual variable and dealing with the Fourier transform of the square root of Π

$$\tilde{\Pi}^{\frac{1}{2}}(p) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx/\hbar} \Pi^{\frac{1}{2}}(x)\,dx. \tag{2.6}$$

The modulus square of $\tilde{\Pi}^{\frac{1}{2}}$ yields the momentum distribution

$$|\tilde{\Pi}^{\frac{1}{2}}(p)|^2 \frac{dp}{\hbar} = \frac{dp}{\pi M\hbar\omega} \exp\left(- \frac{p^2}{M\hbar\omega}\right) \tag{2.7}$$

from which we conclude that the variance of the momentum P is given by

$$\mathrm{Var}\, P = \frac{\hbar}{2} M\omega. \tag{2.8}$$

The uncertainty principle (now confined to the ground state) is expressed in terms of the variances of X and P.

To deal with the coherent state, we modify the drift term defined by (2.2) to correspond to the displaced oscillator

$$\lim_{\Delta\to\infty} \frac{E\{[X(t+\Delta t) - X(t)] \mid X(t) = x\}}{\Delta t} = -i\omega(x-\beta) \tag{2.9}$$

where β is an arbitrary complex number. Then the conditional complex measure density $\Pi(x, t \mid x_0; t_0)$ satisfies the Fokker–Planck equation

$$\frac{\partial \Pi(x,t \mid x_0,t_0)}{\partial t} - i\omega \frac{\partial}{\partial x}[(x-\beta)\Pi(x,t \mid x_0,t_0)] + \frac{i\hbar}{2M} \frac{\partial^2 \Pi(x,t \mid x_0,t_0)}{\partial x^2} \tag{2.10}$$

with the initial condition

$$\Pi(x,t_0 \mid x_0,t_0) = \delta(x-x_0). \tag{2.11}$$

The above equation can be solved and we obtain

$$\Pi(x,t \mid x_0,t_0) = \left(\frac{M\omega}{\pi\hbar(1-e^{2i\omega T})}\right)^{\frac{1}{2}} \exp\left(- \frac{M\omega}{\hbar} \frac{[(x-\beta)\, e^{i\omega T} - (x_0-\beta)]}{e^{2i\omega T} - 1}\right) \tag{2.12}$$

where $T = t - t_0$. The stationary-state solution corresponds to the ground state and is the limit as $T = t - t_0 \to \infty$ of Π

$$\Pi(x,\beta) = \lim \Pi(x,t \mid x_0,t_0) = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(- \frac{M\omega}{\hbar}(x-\beta)^2\right). \tag{2.13}$$

It is worth noting that the limit is still a complex measure density (CMD). Setting $M = 1$, we note that the coherent state wavefunction (Glauber 1963, Sudarshan 1963) can be obtained by taking the square root of the CMD with the identification

$$\alpha = \left(\frac{\omega}{2\hbar}\right)^{\frac{1}{2}} \beta. \tag{2.14}$$

It is worth noting that Sudarshan uses z in the place of α (see Klauder and Sudarshan (1968)). Questions relating to momentum are answered by resorting to a Fourier transform of the square root of the CMD

$$\begin{aligned}\tilde{\Pi}^{\frac{1}{2}}(p,\beta) &= \frac{1}{\sqrt{2\pi}} \int e^{-ipx/\hbar} \Pi^{\frac{1}{2}}(x,\beta)\,dx \\ &= \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{2}} \exp\left(- \left\{ \frac{(p - i\beta\omega)^2}{2\hbar\omega} - \frac{[\beta^2 + (\mathrm{Im}\,\beta)^2]}{2\hbar}\omega \right\}\right). \end{aligned} \tag{2.15}$$

If we take the modulus square, we obtain the results corresponding to the observable modulus measure of the relevant densities

$$\Pi_{11}(x,\beta) = N|\Pi(x,\beta)| = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{\omega}{\hbar}(x - \operatorname{Re}\beta)^2\right) \tag{2.16}$$

$$\Pi_{11}(p,\beta)\,\mathrm{d}p = N|\tilde{\Pi}^{\frac{1}{2}}(p,\beta)|^2 = \frac{\mathrm{d}p}{\hbar}\left(\frac{1}{\pi\omega\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{(p-\omega\operatorname{Im}\beta)^2}{\hbar\omega}\right)\mathrm{d}p \tag{2.17}$$

where N, the normalization constant, is so chosen that

$$\int \Pi_{11}(x,\beta)\,\mathrm{d}x = 1 \tag{2.18}$$

which by Plancharel's theorem ensures that $\Pi_{11}(p,\beta)$ is a conserved momentum density in the sense

$$\int \Pi_{11}(p,\beta)\,\mathrm{d}p = \int \Pi_{11}(x,\beta)\,\mathrm{d}x. \tag{2.19}$$

Using the modulus measure density, we obtain

$$E[X] = \beta_1 = \operatorname{Re}\beta = \left(\frac{2\hbar}{\omega}\right)^{\frac{1}{2}} \operatorname{Re}\alpha \tag{2.20}$$

$$E[P] = \omega\beta_2 = \omega\operatorname{Im}\beta = (2\hbar\omega)^{\frac{1}{2}}\operatorname{Im}\alpha \tag{2.21}$$

$$\operatorname{Var} X = \frac{\hbar}{2\omega} \tag{2.22}$$

$$\operatorname{Var} P = \frac{\hbar\omega}{2}. \tag{2.23}$$

To establish further correspondence with the results of Glauber and Sudarshan, we start with the CMD of the stationary state of the displaced oscillator

$$\Pi_{\mathrm{sty}}(x,\beta) = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar}(x-\beta\sqrt{2})^2\right) \tag{2.24}$$

where the particular choice of β is to make the correspondence with Sudarshan's results complete. We set $\omega = \hbar = 1 = M$ and introduce $\phi_\beta(x)$ by

$$\phi_\beta(x) = N\Pi_{\mathrm{sty}}(x,\beta) \tag{2.25}$$

where N is so chosen to render the total modulus measure unity

$$N = \exp(-2(\operatorname{Im}\beta)^2). \tag{2.26}$$

We next note that the square root $\phi_\beta^{\frac{1}{2}}(x)$ of $\phi_\beta(x)$ belongs to the Hilbert space defined earlier and we can conveniently deal with the projections for they acquire a special meaning in the complex measure theoretic framework. The inner product corresponding to the CMD labelled by β and α is defined by

$$(\phi_\beta^{\frac{1}{2}}, \phi_\alpha^{\frac{1}{2}}) = \int \phi_\beta^{\frac{1}{2}}(x)\overline{\phi_\alpha^{\frac{1}{2}}(x)}\,\mathrm{d}x. \tag{2.27}$$

Evaluating the integral, we obtain

$$(\phi_\beta^{\frac{1}{2}}, \phi_\alpha^{\frac{1}{2}}) = \exp\{\bar{\alpha}\beta - \tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \mathrm{i}\alpha_1\alpha_2 - \mathrm{i}\beta_1\beta_2\} \tag{2.28}$$

which is in agreement with the Glauber–Sudarshan formula except for the phase factors. In the CMTF the phase factor is unique and follows from the fact that all complex measures are normalized to unity. It should be noted that $\phi_\beta^{\frac{1}{2}}$ represents the same state as that denoted by $|\beta\rangle$ in conventional treatment and the left-hand side of (2.28) is nothing but $\langle\alpha \mid \beta\rangle$.

In a similar way we can evaluate the inner product with the Hermite functions $\{\phi_n\}$ which legitimately belong to the Hilbert space in question. Thus we find

$$\langle n \mid \beta\rangle = (\phi_\beta^{\frac{1}{2}}, \phi_n) = \frac{\beta^n}{(n!)^{\frac{1}{2}}} \exp\left(-\frac{|\beta|^2}{2} - \mathrm{i}\beta_1\beta_2\right) \tag{2.29}$$

a relation form which we have

$$\sum_n \langle\beta \mid n\rangle\langle n \mid \alpha\rangle = \exp\left\{\alpha\bar{\beta} - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 - \mathrm{i}\alpha_1\alpha_2 + \mathrm{i}\beta_1\beta_2\right\} \tag{2.30}$$

in agreement with the Glauber–Sudarshan formula except for the phase factors. In the CMTF the phase factor is unique and follows from the normalization (2.25). Generally some arbitrariness arises in the conventional treatment; instead of $(\mathrm{Im}\,\beta)^2$, $(\mathrm{Im}\,\beta)^2 - \mathrm{i}\beta_1\beta_2$ is used. This is due to the fact that the constant term β^2 is replaced by $(1/2)\beta^2 + (1/2)|\beta|^2$ so that the real part completes the square of the argument under the exponential. In the CMTF $\beta^2 \to \beta^2 + (\mathrm{Im}\,\beta)^2$ due to normalization of the modulus measure so that the imaginary part is left intact. The phase factors disappear in the final estimate when we resort to the modulus measure.

Next we show that any CMD f rather than its square root can be expressed in terms of the measure functions corresponding to coherent states. To do this we first form the scalar product of $f^{\frac{1}{2}}$ with $\phi_\alpha^{\frac{1}{2}}$.

$$(f^{\frac{1}{2}}, \phi_\alpha^{\frac{1}{2}}) = \int f^{\frac{1}{2}}(x)\overline{\phi_\alpha^{\frac{1}{2}}}(x)\,\mathrm{d}x. \tag{2.31}$$

To evaluate the integral, we note that $f^{\frac{1}{2}}$ admits an expansion in terms of Hermite functions

$$f^{\frac{1}{2}} = \sum f_n^{\frac{1}{2}}\phi_n(x) \tag{2.32}$$

while†

$$\phi_\alpha^{\frac{1}{2}}(x) = \sum \phi_m(x)\,\mathrm{e}^{-|\alpha|^2/2}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}. \tag{2.33}$$

Thus we have

$$\begin{aligned}(f^{\frac{1}{2}}, \phi_\alpha^{\frac{1}{2}}) &= \sum f_m^{\frac{1}{2}}\,\mathrm{e}^{-|\alpha|^2/2}\frac{\bar{\alpha}^m}{(m!)^{\frac{1}{2}}} \\ &= \tilde{f}^{\frac{1}{2}}(\bar{\alpha})\,\mathrm{e}^{-|\alpha|^2/2}\end{aligned} \tag{2.34}$$

where

$$\tilde{f}^{\frac{1}{2}}(\bar{\alpha}) = \sum_m f_m^{\frac{1}{2}}\frac{\bar{\alpha}^m}{(m!)^{\frac{1}{2}}}. \tag{2.35}$$

It is to be noted that $\tilde{f}^{\frac{1}{2}}(\bar{\alpha})$ is a function of the complex variable $\bar{\alpha}$ associated with the CMD $f(.)$ which is a complex valued function of a real argument. Thus, we are led to an analytic function of the complex variable as in coherence theory; we can now take the

† From now on we dispense with the phase factors in choosing N to be in conformity with conventional treatment.

scalar product with $\phi_\beta^{\frac{1}{2}}$. To do this we can use the operations which are now well defined in the CMTF:

$$\sum_\alpha \langle\beta \mid \alpha\rangle\langle\alpha \mid f^{\frac{1}{2}}\rangle = \sum_\alpha \langle\beta \mid \alpha\rangle \sum_m \langle\alpha \mid m\rangle\langle m \mid f^{\frac{1}{2}}\rangle$$

$$= \frac{1}{\pi}\int \exp\left\{-|\alpha|^2 - \frac{1}{2}|\beta|^2 + \bar{\beta}\alpha\right\}\tilde{f}_{\frac{1}{2}}(\bar{\alpha})\,\mathrm{d}^2\alpha \qquad (2.36)$$

which is the analogue of equation (4.8) of Glauber (1963). We can now expand $\tilde{f}_{\frac{1}{2}}(\bar{\alpha})$ in Taylor series to obtain finally

$$\langle\beta \mid f^{\frac{1}{2}}\rangle = \exp\{-\tfrac{1}{2}|\beta|^2\}\tilde{f}_{\frac{1}{2}}(\bar{\beta}) \qquad (2.37)$$

which leads to a more general projective relation

$$\langle g^{\frac{1}{2}}, f^{\frac{1}{2}}\rangle = \frac{1}{\pi}\int \exp(-|\beta|^2)\,\overline{\tilde{g}^{\frac{1}{2}}(\bar{\beta})}\,f^{\frac{1}{2}}(\bar{\beta})\,\mathrm{d}^2(\bar{\beta}). \qquad (2.38)$$

Thus the correspondence is extensive in all respects. Finally, we observe that it is indeed possible to regard $\alpha(\beta)$ as a (complex measurable) random variable. If we denote by $P(\alpha)$ the complex measure density, the following special choices of $P(\alpha)$, lead to interesting cases known in the literature:

(i) the choice $P(\alpha) = \delta(\alpha - \alpha_0)$ leads to the coherent state (a Poisson distribution for the number of photons);

(ii) the choice $P(\alpha) = (1/\pi\langle n\rangle)\,\mathrm{e}^{(-|\alpha|^2/\langle n\rangle)}$ leads to thermal light (the Bose–Einstein distribution for the number of photons with $\langle n\rangle$ as the mean number).

If the complex measure density in (ii) is centred around α_0, then we obtain the familiar case of the amplitude mixture of coherent and chaotic streams. Thus the notation $P(.)$ is deliberate to bring out the analogy with the P-function of Glauber–Sudarshan.

3. Squeezed coherent state

At the outset we note that in the CMTF the diffusion constant fixes the scale of variance; accordingly we modify the diffusion function by assuming

$$\lim_{\Delta\to\infty} \frac{E\{[X(t+\Delta) - X(t)]^2 \mid X(t) = x\}}{\Delta} = 2D = \frac{\mathrm{i}\hbar\lambda}{M} \qquad (3.1)$$

where λ is an arbitrary positive real number. The drift function is taken to be specified by (2.2). Then the conditional probability measure density Π satisfies (2.10) where $\hbar$ is replaced by $\hbar\lambda$. It is convenient to specify λ by

$$\lambda = \mathrm{e}^{-2s} \qquad (s > 0). \qquad (3.2)$$

Then we find that the stationary-state solution is given by

$$\Pi_{\mathrm{stg}} = \left(\frac{M\omega}{\pi\hbar\lambda}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar\lambda}x^2\right). \qquad (3.3)$$

The expected values are still given by (2.20) and (2.21). The variances are now modified

$$\mathrm{Var}\,X = \frac{\hbar}{2\omega M}\,\mathrm{e}^{-2s} \qquad (3.4)$$

$$\mathrm{Var}\,P = \frac{\hbar\omega M}{2}\,\mathrm{e}^{2s}. \qquad (3.5)$$

It is convenient to introduce the dimensionless variables $\hat{X}$ and $\hat{Y}$

$$\hat{X} = \left(\frac{M\omega}{2\hbar}\right)^{\frac{1}{2}} X \tag{3.6}$$

$$\hat{Y} = (2\hbar M\omega)^{-\frac{1}{2}} P \tag{3.7}$$

so that we have

$$\mathrm{Var}\,\hat{X} = \tfrac{1}{4}\,\mathrm{e}^{-2s} \qquad \mathrm{Var}\,\hat{Y} = \tfrac{1}{4}\,\mathrm{e}^{2s} \tag{3.8}$$

$$\mathrm{Var}\,\hat{X}\,\mathrm{Var}\,\hat{Y} = \tfrac{1}{16}. \tag{3.9}$$

Next, we can consider a more general type of squeezing by modifying (3.1) by

$$2D = \frac{\mathrm{i}\hbar A}{M} \tag{3.10}$$

where $A = A_1 + \mathrm{i}A_2$ is an arbitrary complex parameter with $A_1 > 0$. The stationary solution is now given by

$$\Pi_X(x) = \lim \Pi = \left(\frac{M\omega}{\pi\hbar A}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar A}x^2\right). \tag{3.11}$$

We introduce the normalized measure density $\phi_X(x)$ by

$$\phi_X(x) = N\,\Pi_X(x) \tag{3.12}$$

so that $N \int |\Pi(x)|\,\mathrm{d}x = 1$. The variance follows from the modulus measure density

$$\mathrm{Var}\,X = \frac{1}{2}\frac{|A|^2}{A_1}\frac{\hbar}{\omega M} \tag{3.13}$$

$$\mathrm{Var}\,\hat{X} = \frac{1}{4}\frac{|A|^2}{A_1}. \tag{3.14}$$

The Fourier transform $\phi_P^{\frac{1}{2}}(p)$ of $\phi_X^{\frac{1}{2}}(x)$ is given by

$$\phi_P^{\frac{1}{2}}(p) = \left(\frac{A_1\hbar}{\pi M\omega}\right)^{\frac{1}{4}} \exp\left(-\frac{Ap^2}{2\hbar M\omega}\right) \tag{3.15}$$

so that the modulus measure density of the momentum is given by

$$\phi_{11}(p, A) = |\phi_P^{\frac{1}{2}}(p)|^2\frac{1}{\hbar} = \left(\frac{A_1}{\hbar\pi M\omega}\right)^{\frac{1}{2}} \exp\left(-\frac{A_1 p^2}{\hbar M\omega}\right). \tag{3.16}$$

Thus we have

$$\mathrm{Var}\,P = \frac{1}{2}\frac{\hbar\omega M}{A_1} \qquad \mathrm{Var}\,\hat{Y} = \frac{1}{4}A_1 \tag{3.17}$$

$$\mathrm{Var}\,\hat{X}\,\mathrm{Var}\,\hat{Y} = \frac{1}{16}\left(1 + \frac{A_2^2}{A_1^2}\right) \geqslant \frac{1}{16}. \tag{3.18}$$

When $A_2 = 0$, the product of the variances is a minimum. We can set

$$A = \frac{\mu - \nu}{\mu + \nu} \qquad \mu = \cosh s \qquad \nu = \mathrm{e}^{\mathrm{i}\theta}\sinh s \tag{3.19}$$

to be in conformity with Caves (1981) (see also Loudon and Knight (1987)); then the variances take the form

$$\mathrm{Var}\, X = \frac{\hbar}{2M\omega}\left(\mathrm{e}^{-2s}\cos^2\frac{\theta}{2} + \mathrm{e}^{2s}\sin^2\frac{\theta}{2}\right) \tag{3.20}$$

$$\mathrm{Var}\, P = \frac{M\omega\hbar}{2}\left(\mathrm{e}^{-2s}\sin^2\frac{\theta}{2} + \mathrm{e}^{2s}\cos^2\frac{\theta}{2}\right). \tag{3.21}$$

The most general type of squeezing is obtained by choosing the drift coefficient as specified by (2.9) and the diffusion by (3.10); in this case the stationary complex measure density is given by

$$\Pi(x,\beta) = \left(\frac{M\omega}{\pi\hbar A}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar A}(x-\beta\sqrt{2})^2\right) \tag{3.22}$$

where we have used $\beta\sqrt{2}$ in place of β. As before we define ϕ_β as the normalized density and this leads us to the density function ϕ_γ on setting $M = \hbar = \omega = 1$

$$\phi_\gamma = \left(\frac{1}{2\pi\eta^2}\right)^{\frac{1}{2}} \exp\left(-\left[\frac{x^2}{2\eta^2}\left(1-\mathrm{i}\frac{A_2}{A_1}\right) - \frac{2x\gamma}{\eta} - \gamma^2 - |\gamma|^2\right]\right) \tag{3.23}$$

where η^2 is the natural variance of the coordinate defined by

$$\eta^2 = \frac{1}{2}\frac{|A|^2}{A_1} \tag{3.24}$$

$$\gamma^2 = \frac{\beta^2\bar{A}}{AA_1}. \tag{3.25}$$

The expression (3.23) corresponds to the two photon coherent states discussed by Yuen (1976) and Dodunov *et al* (1980). The states labelled by γ yield states very similar to coherent states and coincide with them for $A_2 = 0$. More generally when $A_2 \neq 0$, the scalar product of any two members of the Hilbert space generated by $\{\phi_\gamma\}$ for two different γ is explicitly given by

$$(\phi_\gamma^{\frac{1}{2}}, \phi_{\gamma'}^{\frac{1}{2}}) = \exp(-\tfrac{1}{2}|\gamma|^2 - |\gamma'|^2 + \bar{\gamma}'\gamma) \tag{3.26}$$

in conformity with equation (3.14) of Yuen (1976). If we reserve the subscript α for the usual coherent states, we also have

$$(\phi_\gamma^{\frac{1}{2}}, \phi_\alpha^{\frac{1}{2}}) = \left(\frac{1}{\mu}\right)^{\frac{1}{2}} \exp\left[-\frac{|\gamma|^2}{2} - \frac{|\bar{\alpha}|^2}{2} - \frac{\bar{\alpha}^2\nu}{2\mu} + \frac{\bar{\alpha}\gamma}{\mu}\left(\frac{\mu-\nu}{\mu-\bar{\nu}}\right)^{\frac{1}{2}} + \frac{\bar{\nu}\gamma^2}{2\mu}\left(\frac{\mu-\nu}{\mu-\bar{\nu}}\right) + \mathrm{i}\theta_0\right] \tag{3.27}$$

where θ_0 is an arbitrary real number. The above formula is in agreement with the one derived by Yuen (1976) provided we identify β of Yuen as $\gamma((\mu-\nu)/(\mu-\bar{\nu}))^{\frac{1}{2}}$.

4. Complex coordinate representation and holomorphic extension

It is very illuminating to extend the formalism developed to the harmonic oscillator with a complex coordinate system. We present the formalism for a general system corresponding to a forced harmonic oscillator since this will be useful in other context like radiation in interaction with matter.

We start with a general Markov process $\{Z(t)\}$ which is complex measurable with the measure of any (measurable) set being absolutely bounded. The drift and diffusion coefficients are specified by

$$\lim_{\Delta\to\infty} \frac{E\{[Z(t+\Delta)-Z(t)] \mid Z(t)=z\}}{\Delta} = -\mathrm{i}wz + B(t) \tag{4.1}$$

$$\lim_{\Delta\to\infty} \frac{E\{[Z(t+\Delta)-Z(t)][\bar{Z}(t+\Delta)-\bar{Z}(t)] \mid Z(t)=t\}}{\Delta} = 4D. \tag{4.2}$$

The drift and diffusion coefficients correspond to the Langevin equation

$$\ddot{z} = -\omega^2 z + f(t) \tag{4.3}$$

and $B(t)$ is related to $f(t)$ by

$$B(t) = \int_0^t f(u)\,\mathrm{e}^{\mathrm{i}\omega(t-u)}\,\mathrm{d}u. \tag{4.4}$$

If $\Pi(z|z_0, t)$ is the complex measure density, then Π satisfies the Fokker–Planck equation

$$\frac{\partial\Pi(z \mid z_0, t)}{\partial t} = -\frac{\partial}{\partial z}[\mathrm{i}\omega z + B(t)]\Pi - \frac{\partial}{\partial z}[-\mathrm{i}\omega\bar{z} + \bar{B}(t)]\Pi + 4D\frac{\partial^2\Pi}{\partial z\partial\bar{z}}. \tag{4.5}$$

It is shown in the appendix that (4.5) can be solved with its solution given by (A.8). Reverting to real variables, we find

$$\Pi(z \mid z_0, t) = \frac{M\omega\,\mathrm{e}^{2\mathrm{i}\omega t}}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t}-1)} \exp\left\{-\frac{M\omega}{\hbar(\mathrm{e}^{2\mathrm{i}\omega t}-1)}\left[\left\{x\,\mathrm{e}^{\mathrm{i}\omega t} - x_0 - \int_0^t \frac{B+\bar{B}}{2}\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s\right\}^2 + \left\{y\,\mathrm{e}^{\mathrm{i}\omega t} - y_0 - \int_0^t \frac{B-\bar{B}}{2\mathrm{i}}\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s\right\}^2\right]\right\} \tag{4.6}$$

where $2D$ is chosen to be $(\mathrm{i}\hbar/2M)$ and $z = x + \mathrm{i}y$. From the general result given earlier we can deduce that the stationary-state solution of a free harmonic oscillator ($B(t) = 0$) corresponding to the ground state is

$$\Pi^H_{\mathrm{sty}}(z) = \frac{M\omega}{\pi\hbar}\exp\left(-\frac{M\omega}{\hbar}(x^2+y^2)\right). \tag{4.7}$$

Likewise we can independently deduce that the ground-state (stationary-state) solution of a displaced oscillator with displacement β is given by

$$\Pi^H_{\mathrm{sty}} = \frac{M\omega}{\pi\hbar}\exp\left(-\frac{M\omega}{\hbar}(z-\beta)(\bar{z}-\bar{\beta})\right). \tag{4.8}$$

Before we discuss the implication of these results, we also present a solution for Π when initially the oscillator is constrained to be in the ground state rather than with coordinates constrained at z_0. This is simply obtained by multiplying Π by the ground-state measure density evaluated at z_0 and then integrating over x_0 and y_0

$$\Pi^H_{\mathrm{gd}}(z,t) = \frac{M\omega}{\pi\hbar}\exp\left\{-\frac{M\omega}{\hbar}\left[\left(z - \int_0^t B(s)\,\mathrm{e}^{-\mathrm{i}\omega(t-s)}\,\mathrm{d}s\right)\left(\bar{z} - \int_0^t \bar{B}(s)\,\mathrm{e}^{-\mathrm{i}\omega(t-s)}\,\mathrm{d}s\right)\right]\right\}. \tag{4.9}$$

This formula is quite useful in developing a formalism to deal with the problem of the interaction of radiation with matter on lines parallel to the path integral development of Feynman and Hibbs (1965).

At the outset we note that the ground-state solution of the free harmonic oscillator given by (4.7) is the holomorphic extension of the complex measure density (2.5). The random variable $Z = X + \mathrm{i}Y$ has the measure density given by (4.5) and the variances are now given by

$$\mathrm{Var}\, X = \mathrm{Var}\, Y = \frac{\hbar}{2M\omega}. \tag{4.10}$$

If we set

$$X = Q \text{ (coordinate)} \qquad Y = \frac{P}{M\omega} \tag{4.11}$$

$$\hat{X} = Q\left(\frac{\omega}{2\hbar}\right)^{\frac{1}{2}} \qquad \hat{Y} = \frac{P}{(2\hbar\omega)^{\frac{1}{2}}} \tag{4.12}$$

we find

$$\Pi^H(\hat{z}) = \frac{2}{\pi}\exp(-2|\hat{z}|^2). \tag{4.13}$$

Thus, $\Pi^H(\hat{z})$ can be interpreted to be the Wigner distribution. From now on we write z in the place of $\hat{z}$ (misuse of notation). If we make the transformation

$$z = \mu\zeta + \nu\bar{\zeta} \qquad \mu = \cosh s \qquad \nu = \mathrm{e}^{\mathrm{i}\theta}\sinh s \tag{4.14}$$

then by a repeated misuse of the notation, we obtain the Wigner distribution for the squeezed state

$$\begin{aligned}\Pi^H(z) &= \frac{2}{\pi}\exp(-2|\mu z + \nu\bar{z}|^2)\\ &= \frac{2}{\pi}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} + \frac{y^2}{\sigma_Y^2} - \frac{2\rho_{XY}xy}{\sigma_X\sigma_Y}\right)\right\}\end{aligned} \tag{4.15}$$

where

$$\begin{aligned}\sigma_X^2 &= \frac{1}{4}(\mu-\nu)(\mu-\bar{\nu}) = \frac{1}{4}\left(\mathrm{e}^{2s}\sin^2\frac{\theta}{2} + \mathrm{e}^{-2s}\cos^2\frac{\theta}{2}\right)\\ \sigma_Y^2 &= \frac{1}{4}(\mu+\nu)(\mu+\bar{\nu}) = \frac{1}{4}\left(\mathrm{e}^{2s}\cos^2\frac{\theta}{2} + \mathrm{e}^{-2s}\sin^2\frac{\theta}{2}\right)\\ \rho &= -\frac{\sinh 2s\sin\theta}{4\sigma_X\sigma_Y}.\end{aligned} \tag{4.16}$$

The expression (4.15) has been derived by Dodunov *et al* (1980) who preferred to identify the state as a correlated state. The results relating to the variances and other properties obtained by Caves (1981), Yuen (1976) and Loudon and Knight (1988) follow.

A still more general state can be obtained by starting from (4.8) and then applying a 'squeezing' transformation (4.14); then we obtain, again retaining z for the transformed variable

$$\Pi^H = \frac{2}{\pi}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mathrm{Re}\,\alpha)^2}{\sigma_X^2} + \frac{(y-\mathrm{Im}\,\alpha)^2}{\sigma_Y^2} - \frac{2\rho(x-\mathrm{Re}\,\alpha)(y-\mathrm{Im}\,\alpha)}{\sigma_X\sigma_Y}\right\}\right] \tag{4.17}$$

where $\beta = \mu\alpha + \nu\bar{\alpha}$.

The holomorphic CMD of the displaced oscillator given by (4.8) admits an elegant expansion in terms of complex Hermite functions (Hida 1980). We replace β by $\alpha\sqrt{2}$ and express Π_H the holomorphic CMD, or rather its square root as

$$\Pi_H^{\frac{1}{2}} = \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}(z-\alpha\sqrt{2})(\bar{z}-\bar{\alpha}\sqrt{2})\right)$$
$$= \left(\frac{1}{\pi}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}|z|^2 - \frac{1}{2}|\alpha|^2\right] \sum \frac{H_{mn}(z,\bar{z})}{(m!)^{\frac{1}{2}}(n!)^{\frac{1}{2}}} \left(\frac{\bar{\alpha}}{\sqrt{2}}\right)^m \left(\frac{\alpha}{\sqrt{2}}\right)^n. \tag{4.18}$$

This corresponds to the coherent state labelled by α, the complex Hermite functional representation being a typical one. We can now obtain the projection of $\Pi_H^{\frac{1}{2}}(\alpha)$ on $\Pi_H^{\frac{1}{2}}(\beta)$; thus, we have

$$(\Pi_H^{\frac{1}{2}}(\alpha), \Pi_H^{\frac{1}{2}}(\beta)) = \iint \frac{1}{\pi}\left\{ \exp\left[-\frac{1}{2}(z-\alpha\sqrt{2})(\bar{z}-\bar{\alpha}\sqrt{2})\right]\right.$$
$$\left.\times \exp\left[-\frac{1}{2}(z-\beta\sqrt{2})(\bar{z}-\bar{\beta}\sqrt{2})\right]\right\} \mathrm{d}x\,\mathrm{d}y$$
$$= \exp\left[-\frac{1}{2}|\alpha-\beta|^2\right] \tag{4.19}$$

which is the generalization of the formula for $\langle\beta \mid \alpha\rangle$ in the coherent state representation (2.30).

Next we explore the possibility of expanding any holomorphic extension of the CMD in terms of the coherent state; this runs parallel to the development discussed in section 2. Thus if f is the extension of any CMD then we can define the projection by

$$(f^{\frac{1}{2}}, \Pi_H^{\frac{1}{2}}) = \iint \overline{\Pi_H^{\frac{1}{2}}(\alpha)} f^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}y. \tag{4.20}$$

We expand $f^{\frac{1}{2}}$ in terms of Hermite functions

$$f^{\frac{1}{2}} = \sum f_{mn}^{\frac{1}{2}} \frac{H_{mn}(z,\bar{z})}{(m!n!)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}|z|^2\right). \tag{4.21}$$

Using the expansion (4.18), we have

$$(f^{\frac{1}{2}}, \Pi_H^{\frac{1}{2}}) = \tilde{f}^{\frac{1}{2}}(\alpha,\bar{\alpha})\,\mathrm{e}^{-|\alpha|^2/2} \tag{4.22}$$

where

$$\tilde{f}_{1/2}(\bar{\alpha},\alpha) = \sum f_{mn}^{\frac{1}{2}} \left(\frac{\alpha}{\sqrt{2}}\right)^m \left(\frac{\bar{\alpha}}{\sqrt{2}}\right)^n. \tag{4.23}$$

Thus $\tilde{f}_{1/2}$ is a function of α and $\bar{\alpha}$ satisfying positive definiteness

$$\overline{\tilde{f}_{1/2}(\alpha,\bar{\alpha})} = \tilde{f}_{1/2}(\alpha,\bar{\alpha}). \tag{4.24}$$

It is to be noted that there still remains the problem of arriving at an holomorphic extension in cases other than Gaussian. It is in this context the generalization of the P-function introduced earlier assumes significance.

At this stage some general remarks are in order. The amplitude function corresponding to ϕ_γ as given by (3.23) as well as the Wigner distribution (4.15) was derived by Dodunov *et al* (1980) using the conventional Hilbert space approach. The coefficient of $(x^2/2\eta^2)$ under the exponent in (3.23) is taken to be the complex parameter $1-(\mathrm{i}r/(1-r^2)^{\frac{1}{2}})$ and r is identified as the correlation coefficient of the conjugate variables. Dodunov *et al* attempt to explain away the apparent conflict with the complementarity principle by the statement that a knowledge of the existence of the correlation coefficient implies an enhanced (uncertainty) value of the product of the variances. This should be contrasted with the approach of Caves (1981), Yuen (1976) and Loudon and Knight (1987) who always avoid any statement on the existence or otherwise of the correlation coefficient. Now the results provided in sections 2 and 3 also avoid any reference to the joint distribution of coordinates and momentum; in fact the results presented in sections 2 and 3 are in strict conformity with the complementarity principle; the results relating to the coordinate are derived first and those relating to momentum are then derived by the extended principle of duality. Although the expression for ϕ_γ given by (3.23) is in agreement with the one derived by Dodunov, the CMFT cannot determine the constant r identified as A_2/A_1, even by a thought experiment. When the physically observable modulus measure transformation is made, r slips out of the picture. Thus the CMTF avoids any reference to the joint distribution/correlation between the coordinate and momentum. However, the situation gets drastically altered when the complex measure density for the complex coordinate is introduced; identifying the complex measure density as the Wigner distribution tacitly assumes the existence of the joint measure density although in a complex measure theoretic sense. Thus the Fokker–Planck equation for the measure density has a transparency, albeit a bit embarassing. If complementarity has to be preserved absolutely in its best form, it is prudent to avoid the use of the results presented in this section; the results presented in sections 2 and 3 are viable enough to describe all the situations we are likely to encounter in fundamental physics.

5. Summary and conclusions

In this paper we have worked within the measure theoretic framework to discuss the basic formulation of a coherent state and squeezed coherent state with the harmonic oscillator as the basic building block. The motion of the quantum harmonic oscillator is identified as a complex measurable stochastic process with a Markov property; an appropriate choice of the drift and diffusion function leads to the correct complex measure density in configuration space. The diffusion function which fixes the scale of the variance plays a key role; different choices of the diffusion function lead to different types of complex measure density. The complex measure density function or rather its square root by virtue of the constraints imposed generates a suitable Hilbert space framework. We have shown that the CMTF is versatile enough to describe diverse situations and yield many of the properties of the coherent states and squeezed coherent states. A general harmonic oscillator with complex coordinates has also been discussed; in particular we have shown that this leads to a Fokker–Planck equation whose solution can be identified with the Wigner distribution in special cases. The logical problems that arise by the introduction of complex coordinates along with their identification as conjugate variables have also been discussed. While the complex coordinate representation and the holomorphic extension of the CMD may appear to be powerful, it should be conceded that it has a limited role in the description of quantum phenomena in view of its apparent conflict with complementary principles.

Acknowledgments

Part of this work was finalized at the Centre for Particle Physics, University of Texas in Austin during the spring of 1996. The author acknowledges the hospitality extended to him by Professor E C G Sudarshan to whom he is also grateful for many useful discussions.

Appendix

Our starting point is the Fokker–Planck equation (4.3) satisfied by Π. We convert the equation using real variables. Thus the equation becomes

$$\frac{\partial \Pi}{\partial t} = \frac{\partial \Pi}{\partial x}\left(\mathrm{i}\omega x - \frac{B+\bar{B}}{2}\right) + 2\mathrm{i}\omega\Pi + \frac{\partial \Pi}{\partial y}\left(\mathrm{i}\omega y - \frac{B-\bar{B}}{2\mathrm{i}}\right) + \frac{D}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Pi. \quad \text{(A.1)}$$

By introducing the characteristic coordinates (ξ, η) by

$$\xi = x\,\mathrm{e}^{\mathrm{i}\omega t} - \frac{1}{2}\int_0^t (B+\bar{B})\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s \quad \text{(A.2)}$$

$$\eta = y\,\mathrm{e}^{\mathrm{i}\omega t} - \frac{1}{2}\mathrm{i}\int_0^t (B-\bar{B})\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s \quad \text{(A.3)}$$

and setting

$$\Pi(x, y, t) = \mathrm{e}^{2\mathrm{i}\omega t}\rho(\xi, \eta, t) \quad \text{(A.4)}$$

we obtain

$$\frac{\partial \rho}{\partial t} = D\,\mathrm{e}^{2\mathrm{i}\omega t}\left(\frac{\partial^2 \rho}{\partial \xi^2} + \frac{\partial^2 \rho}{\partial \eta^2}\right). \quad \text{(A.5)}$$

The solution for the initial condition

$$\Pi(x, y, 0) = \delta(x - x_0)\delta(y - y_0) \quad \text{(A.6)}$$

is given by

$$\rho = \frac{M\omega}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\exp\left[-\frac{M\omega}{\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}[(\xi - \xi_0)^2 + (\eta - \eta_0)^2]\right]. \quad \text{(A.7)}$$

We thus have

$$\Pi(z, z_0, t) = \frac{M\omega\,\mathrm{e}^{2\mathrm{i}\omega t}}{\pi\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\exp\left[-\frac{M\omega}{\hbar(\mathrm{e}^{2\mathrm{i}\omega t} - 1)}\left(z\,\mathrm{e}^{\mathrm{i}\omega t} - z_0 - \int_0^t B(s)\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s\right)\right.$$
$$\left.\times\left(\bar{z}\,\mathrm{e}^{\mathrm{i}\omega t} - \bar{z}_0 - \int_0^t \bar{B}(s)\,\mathrm{e}^{\mathrm{i}\omega s}\,\mathrm{d}s\right)\right]. \quad \text{(A.8)}$$

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